SPACES OF CONTINUOUS LINEAR MAPPINGS

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NORMED SPACE

Definition. Given a linear space X over ℂ (or ℝ.), a mapping
‖. ‖: X → ℝ is a norm for X if it satisfies the following properties:
For all $x \in$ X,
(i) $\|x\| \geq 0$,
(ii) $\|x\| = 0$ if and only if $x = 0$,
(iii) $\|\lambda x\| = |\lambda|\|x\|$ for all scalar λ,
and for all x, y ∈ X,
(iv) $\|x + y\| \leq \|x\| + \|y\|$.

NORMED LINEAR SPACE STRUCTURE

Definition. Given a normed linear space $(X, \|.\|)$ and a linear subspace Y of X, it is clear that the restriction of the norm $\|.\|$ to Y is also a norm for Y. The restriction is denoted $\|.\|_Y$ and (Y, $\|.\|_Y$) is a *normed linear subspace* of $(X, \|.\|)$.

FINITE DIMENSIONAL NORMED LINEAR SPACES

Proposition. An n-dimensional Euclidean space (Unitary space)
(i) is complete and
(ii) has the property that a subset is compact if and only if it is closed and bounded.
Corollary. Every finite dimensional normed linear space is complete.
Corollary. Every linear mapping of a finite dimensional normed linear space into a normed linear space is continuous.

INNER PRODUCT SPACES

Definition. Given a linear space X over ℂ, a mapping (. , .): $X \times X$ → ℂ is an inner product on X if it satisfies the following properties:
For all $x, y, z \in X$
(i) $(x + y, z) = (x, z) + (y, z)$
(ii) $(\lambda x, y) = \lambda(x, y)$ for all $\lambda \in \mathbb{C}$
(iii) $(y, x) = \overline{(x, y)}$
(iv) $(x, x) \geq 0$
(v) $(x, x) = 0$ if and only if $x = 0$.
A linear space X with an inner product (. , .) is called an *inner product space* (or a *pre − Hilbert space*) and is sometimes denoted formally as a pair $(X, (. , .))$.
Definition. An inner product space which is complete as a normed linear space is called a *Hilbert space.*
Example. Unitary n-space (Euclidean n-space) is an inner product space with inner product (. , .) defined on $\mathbb{C}^n (or\ \mathbb{R}^n)$
for $x \equiv (\lambda_1, \lambda_2, \ldots, \lambda_n, \ldots)$ and $y \equiv (\mu_1, \mu_2, \ldots, \mu_n)$ *by*

$$(x, y) = \sum_{k=1}^{n} \lambda_k \overline{\mu_k}.$$

It is quite clear that all the inner product properties (i)-(v) are satisfied. The norm generated by this inner product is the Unitary (Euclidean) norm,

$$\|x\|_2 = \sqrt{\sum_{k=1}^{n} |\lambda_k|^2}\ .$$

We deduce from the inner product structure that the Cauchy-Schwarz inequality holds and applying this inequality to $(|\lambda_1|, |\lambda_2|, \ldots, |\lambda_n|)$ and $(|\mu_1|, |\mu_2|, \ldots, |\mu_n|) \in \mathbb{R}^n$,
we have for $(\lambda_1, \lambda_2, \ldots, \lambda_n, \ldots)$ and $(\mu_1, \mu_2, \ldots, \mu_n) \in \mathbb{C}^n$ that

$$\sum_{k=1}^{n} |\lambda_k \overline{\mu_k}| = \sum_{k=1}^{n} |\lambda_k||\mu_k| \leq \sqrt{\sum_{k=1}^{n} |\lambda_k|^2}\sqrt{\sum_{k=1}^{n} |\mu_k|^2}\ .$$

Definition. Given an inner product space X, for $x, y \in X$ we say that x is orthogonal to y if $(x, y) = 0$ and we write $x \perp y$. Given subset M of X we say that x is *orthogonal* to M if x is orthogonal to every element of M and we write $x \perp M$.

SPACES OF CONTINUOUS LINEAR MAPPINGS

Definitions.
(i) Given linear spaces X and Y over the same scalar field, the set $\mathcal{L}(X, Y)$ of linear mappings of X into Y is a linear space under pointwise definition of addition and multiplication by a scalar; that is, for $T, S \in \mathcal{L}(X, Y)$,
$T + S: X \to Y$ is defined by $(T + S)(x) = Tx + Sx$
and $aT: X \to Y$ is defined by $(aT)(x) = aTx$.
There is no difficulty in verifying that $T + S$ and $aT \in \mathcal{L}(X, Y)$ and that the linear space properties hold.
(ii) Given normed linear spaces $(X, \|.\|)$ and $(Y, \|.\|')$ over the same scalar field, the set $\beta(X, Y)$ of continuous linear mappings of X into Y is a linear subspace of $\mathcal{L}(X, Y)$. Closure under the linear operations follows from the closure of continuity under these operations.
These spaces are in fact generalisations of the finite dimensional case.
Example. When $X = \mathbb{R}^n (or\ \mathbb{C}^n)$ and $Y = \mathbb{R}^m (or\ \mathbb{C}^m)$ then $\beta(X, Y)$ is isomorphic to the linear space M_{mxn} of $m \times n$ matrices with entries from $\mathbb{R} (or\ \mathbb{C})$.

DUAL SPACES

Given normed linear spaces $(X, \|.\|)$ and $(Y, \|.\|')$ our first special case arises when we take the range space Y as the scalar field of the domain space X. We then have the normed linear space of continuous linear functionals on $(X, \|.\|)$.
Definitions.
(i) Given a linear space X over ℂ $(or$ ℝ), the algebraic dual (or algebraic conjugate) space is the linear space $\mathcal{L}(X, \mathbb{C})$(or $\mathcal{L}(X, \mathbb{R})$), usually denoted by $X^{\#}$.
(ii) Given a normed linear space $(X, \|.\|)$ over ℂ $(or$ ℝ), the dual (or conjugate) space is the normed linear space $(\beta(X, \mathbb{C}), \|.\|)$ (or $(\beta(X, \mathbb{R}), \|.\|)$) usually denoted by $(X, \|.\|)^*$. When we are thinking of the dual as a linear space we denote it by X^* and as a normed linear space with its norm, by $(X^*, \|.\|)$. The norm on X^* is given by

$$\|f\| = \sup\{|f(x)|: \|x\| \leq 1\}$$

Corollary. Whether a normed linear space $(X, \|.\|)$ is complete or not, its dual space $(X^*, \|.\|)$ is always complete.

OPERATOR ALGEBRAS

Given normed linear spaces $(X, \|.\|)$ and $(Y, \|.\|')$ our other special case arises when we take the range space $(Y, \|.\|')$ to be the same as the domain space $(X, \|.\|)$. We then have the normed linear space of continuous linear operators on $(X, \|.\|)$.
Definitions. Given an algebra A over $\mathbb{C}(or\ \mathbb{R})$, a norm $\|.\|$ on A is said to be an *algebra norm* if it satisfies the additional norm property:
For all $x, y \in A$, $\|xy\| \leq \|x\|\|y\|$, (the submultiplicative inequality).
The pair $(A, \|.\|)$ İs called a normed algebra. A normed algebra which is complete as a normed linear space is called a complete normed algebra(or a Banach algebra).
Different norms can be assigned to the same algebra A giving rise to different normed algebras.

THE SHAPE OF THE DUAL

Given a n-dimensional linear space X_n over $\mathbb{C}(or\ \mathbb{R})$ with basis $\{e_1, e_2, \ldots, e_n\}$ that the algebraic dual $X_n^{\#}$ is also a linear space with basis $\{f_1, f_2, \ldots, f_n\}$ dual to $\{e_1, e_2, \ldots, e_n\}$ where

$$f_k(e_j) = 1 \quad \text{when } j = k$$
$$= 0 \quad j \neq k$$

Furthermore, since $f = \sum_{k=1}^{n} f(e_k) f_k$,$X_n^{\#}$ is isomorphic to $\mathbb{C}^n (or\ \mathbb{R}^n)$ under the mapping

$$f \mapsto (f(e_1), f(e_2), \ldots, f(l_n))\ .$$

and every linear functional f on X_n is of the form

$$f(x) = \sum_{k=1}^{n} \lambda_k f(e_k) \quad \text{where} \quad x \equiv \sum_{k=1}^{n} \lambda_k e_k\ .$$

Being an isomorphism onto $\mathbb{C}^n$ implies that every linear functional f on X_n has the form

$$f(x) = \sum_{k=1}^{n} \lambda_k \bar{a}_k \text{ for some } (a_1, a_{2,\ldots,} a_n) \in \mathbb{C}^n \text{ (a1,a2,....,an) E } \mathbb{C}^n$$
$$\text{where } x \equiv \sum_{k=1}^{n} \lambda_k e_k\ .$$

PREFERENCE

Linear Functional Analysis, Springer Undergraduate Mathematics Series, Bryan P.Rynne and Martin A.Youngson(2000).
Introduction to the Analysis of Normed Linear Spaces J. R. Giles, Department of Mathemtics, University of Newcastle, Australia,(2000).